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A MORE ACCURATE SOLUTION TO THE ELASTIC-PLASTIC PROBLEM OF PRESSURIZED THICK-WALLED CYLINDERS

PETER C. T. CHEN

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US ARMY ARMAMENT RESEARCH AND DEVELOPMENT CENTER
LARGE CALIBER WEAPON SYSTEMS LABORATORY
BENÉT WEAPONS LABORATORY
WATERVLIET N.Y. 12189

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A new method has been developed for solving the partially plastic problems of thick-walled cylinders made of strain-hardening or ideally-plastic materials subjected to any combination of internal pressure, external pressure, and end loads. The incremental strains are chosen as the basic unknowns in the finite-difference formulation. The incremental sizes of the applied loading are determined automatically and no iteration is needed. Complete solutions

(CONT'D ON REVERSE)

20. ABSTRACT (CONT'D)

for the stresses, strains, and displacement have been obtained and all numerical results are very accurate. This approach is also efficient and simple, yet quite general, when compared with many solutions in the literature.

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INTRODUCTION

The partially plastic problem of pressurized thick-walled cylinder is of practical importance to pressure vessels and the autofrettage process of gun barrels. Many solutions for this problem have been reported (refs 1-7). For thick tubes under very high pressure operation, the elastic-plastic material model should be represented by the von Mises' yield criterion, Prandtl-Reuss' incremental stress-strain laws, the strain-hardening, and compressibility (ref 8). However, a closed-form solution exists only in the plane-strain case neglecting strain-hardening and compressibility.

For the generalized plane-strain problems considered here, numerical solutions were reported by the finite-difference method (refs 4,7) and finite-element method (ref 5). The incremental displacements were used as the basic unknowns and a displacement function was assumed in the finite-element method (ref 5). The incremental stresses and strains were used in Reference 4 as the basic unknowns, but only the incremental strains were used in Reference 7. The spatial discretization used in References 4 and 7 was based on the forward difference scheme and a fixed sequence of incremental loading was used.

In this report, a new method is developed and more accurate numerical results are obtained. The incremental strains are chosen as the basic unknowns in the finite-difference formulation. Both strain-hardening and ideally-plastic materials can be considered. The spatial discretization is based on the central difference scheme and the incremental sizes of the applied loading are determined automatically in the program. The incremental

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References are listed at the end of this report.

results are calculated directly and no iteration is needed. The convergence of the approach will be discussed and more accurate results will be reported.

BASIC EQUATIONS

Assuming small strain and no body forces in the axisymmetric state of generalized plane-strain, the radial and tangential stresses, σ_r and σ_θ, must satisfy the equilibrium equation,

$$r(\partial\sigma_r/\partial r) = \sigma_\theta - \sigma_r \tag{1}$$

and the corresponding strains, ε_r and ε_θ, are given in terms of the radial displacement, u, by

$$\varepsilon_r = \partial u/\partial r \quad , \quad \varepsilon_\theta = u/r \tag{2}$$

It follows that the strains must satisfy the equation of compatibility

$$r(\partial\varepsilon_\theta/\partial r) = \varepsilon_r - \varepsilon_\theta \tag{3}$$

If the material is assumed to be elastic-plastic, obeying the Mises' yield criterion, the Prandtl-Reuss flow theory, and the isotropic hardening law, the stress-strain relations are (ref 1):

$$d\varepsilon_i' = d\sigma_i'/2G + (3/2)\sigma_i' d\sigma/(\sigma H') \tag{4}$$

$$d\sigma > 0 \quad \text{for } i = r,\theta,z$$

$$d\varepsilon_m = E^{-1}(1-2\nu)d\sigma_m \tag{5}$$

where E, ν are Young's modulus, Poisson's ratio, respectively,

$$2G = E/(1+\nu),$$

$$\varepsilon_m = (\varepsilon_r+\varepsilon_\theta+\varepsilon_z)/3 \quad , \quad \varepsilon_i' = \varepsilon_i - \varepsilon_m$$

$$\sigma_m = (\sigma_r+\sigma_\theta+\sigma_z)/3 \quad , \quad \sigma_i' = \sigma_i - \sigma_m$$

$$\sigma = (1/\sqrt{2})[(\sigma_r-\sigma_\theta)^2 + (\sigma_\theta-\sigma_z)^2 + (\sigma_z-\sigma_r)^2]^{1/2} > \sigma_o \tag{6}$$

and σ_o is the yield stress in simple tension or compression. For a strain-hardening material, H' is the slope of the effective stress/plastic strain

curve

$$\sigma = H(\int d\varepsilon^p) \tag{7}$$

For an ideally-plastic material ($H' = 0$), the quantity $(3/2)d\sigma/(\sigma H')$ is replaced by $d\lambda$, a positive factor of proportionality. When $\sigma < \sigma_o$ or $d\sigma < 0$, the state of stress is elastic and the second term in Eq. (4) disappears. Following Yamada et al (ref 9), Eqs. (4) and (5) can be rewritten in an incremental form

$$d\sigma_i = d_{ij}d\varepsilon_j \quad \text{for } i,j = r,\theta,z \tag{8}$$

and

$$d_{ij} = 2G[\nu/(1-2\nu) + \delta_{ij} - \sigma_i'\sigma_j'/S]$$

where

$$S = \frac{2}{3}\left(1 + \frac{1}{3}H'/G\right)\sigma^2 \quad , \quad H'/E = \omega/(1-\omega) \tag{9}$$

ωE is the slope of the effective stress-strain curve, and δ_{ij} is the Kronecker delta.

This form was used in the finite-element formulation for solving elastic-plastic thick-walled tube problems (ref 5). In the following section, the incremental stress-strain matrix will be used in the finite-difference formulation.

FINITE-DIFFERENCE FORMULATION

Consider a thick-walled cylinder of inner radius a and external radius b. The tube is subjected to inner pressure p, external pressure q, and end force f. The elastic solution for this problem is well-known and the pressure p*, q*, or f* required to cause initial yielding can be determined by using the Mises' yield criterion. For loading beyond the elastic limit, an incremental approach of the finite-difference formulation is used. The cross-section of

the tube is divided into n rings with $r_1=a, r_2, \ldots, r_k=\rho, \ldots, r_{n+1}=b$ where ρ is the radius of the elastic-plastic interface. At the beginning of each increment of loading, the distribution of displacements, strains, and stresses are assumed to be known and we want to determine Δu, $\Delta\varepsilon_r$, $\Delta\varepsilon_\theta$, $\Delta\varepsilon_z$, $\Delta\sigma_r$, $\Delta\sigma_\theta$, $\Delta\sigma_z$ at all grid points. Since the incremental stresses are related to the incremental strains by the incremental form (Eq. (8)) and $\Delta u = r\Delta\varepsilon_\theta$, there exists only three unknowns at each station that have to be determined for each increment of loading. Accounting for the fact that the axial strain ε_z is independent of r, the unknown variables in the present formulation are $(\Delta\varepsilon_\theta)_i$, $(\Delta\varepsilon_r)_i$, for $i = 1,2,\ldots,n,n+1$, and $\Delta\varepsilon_z$.

The equation of equilibrium (1) and the equation of compatibility (3) are valid for both the elastic and the plastic regions of a thick-walled tube. A first-order-correct finite-difference analog of these two equations at $i = 1,\ldots,n$ has been given in References 4 and 7. Other finite-difference forms can be written. In this report, the difference equations given below are second-order-correct. The equation of compatibility (3) and equation of equilibrium (1) are replaced, respectively, by

$$c_{1i}(\Delta\varepsilon_\theta)_i + c_{2i}(\Delta\varepsilon_r)_i + c_{3i}(\Delta\varepsilon_\theta)_{i+1} + c_{4i}(\Delta\varepsilon_r)_{i+1} = c_{5i} \tag{10}$$

and

$$c_{1i}(\Delta\sigma_r)_i + c_{2i}(\Delta\sigma_\theta)_i + c_{4i}(\Delta\sigma_r)_{i+1} + c_{4i}(\Delta\sigma_\theta)_{i+1} = c_{6i} \tag{11}$$

where

$$c_{1i} = -\frac{3}{2} + \frac{1}{2}\gamma_i \ , \quad c_{2i} = \frac{1}{2} - \frac{1}{2}\gamma_i$$

$$c_{3i} = \frac{3}{2} - \frac{1}{2}\gamma_i \ , \quad c_{4i} = -\frac{1}{2} + \frac{1}{2}\gamma_i$$

$$c_{5i} = -c_{1i}(\varepsilon_\theta)_i - c_{2i}(\varepsilon_r)_i - c_{3i}(\varepsilon_\theta)_{i+1} - c_{4i}(\varepsilon_r)_{i+1}$$

$$c_{6i} = -c_{1i}(\sigma_r)_i - c_{2i}(\sigma_\theta)_i - c_{3i}(\sigma_r)_{i+1} - c_{4i}(\sigma_\theta)_{i+1}$$

and

$$\gamma_i = r_{i+1}/r_i \tag{12}$$

With the aid of the incremental stress-strain relations (8), Eq. (11) can be written as

$$c_{7i}(\Delta\varepsilon_\theta)_i + c_{8i}(\Delta\varepsilon_r)_i + c_{9i}(\Delta\varepsilon_\theta)_{i+1} + c_{10i}(\Delta\varepsilon_r)_{i+1} + c_{11i}(\Delta\varepsilon_z) = c_{6i} \tag{13}$$

where

$$c_{7i} = c_{1i}(d_{12})_i + c_{2i}(d_{22})_i \quad , \quad c_{8i} = c_{1i}(d_{11})_i + c_{2i}(d_{21})_i$$

$$c_{9i} = c_{3i}(d_{12})_{i+1} + c_{4i}(d_{22})_{i+1} \quad , \quad c_{10i} = c_{3i}(d_{11})_{i+1} + c_{4i}(d_{22})_{i+1}$$

$$c_{11i} = c_{1i}(d_{13})_i + c_{2i}(d_{23})_i + c_{3i}(d_{13})_{i+1} + c_{4i}(d_{23})_{i+1} \tag{14}$$

The boundary conditions for the problem are

$$\Delta\sigma_r(a,t) = -\Delta p \quad , \quad \Delta\sigma_r(b,t) = -\Delta q$$

$$\pi \sum_{i=1}^{n} [r_i(\Delta\sigma_z)_i + r_{i+1}(\Delta\sigma_z)_{i+1}](r_{i+1}-r_i) = \mu\pi a^2\Delta p + \Delta f \tag{15}$$

where μ is zero for open-end tubes, and one for closed-end tubes. Using the incremental relations (8), we rewrite Eq. (15) as

$$(d_{12})_1(\Delta\varepsilon_\theta)_1 + (d_{11})_1 (\Delta\varepsilon_r)_1 + (d_{13})_1\Delta\varepsilon_z = -\Delta p \tag{16}$$

$$(d_{12})_{n+1}(\Delta\varepsilon_\theta)_{n+1} + (d_{11})_{n+1}(\Delta\varepsilon_r)_{n+1} + (d_{13})_{n+1}\Delta\varepsilon_z = -\Delta q \tag{17}$$

and

$$\sum_{i=1}^{n} [c_{12i}(\Delta\varepsilon_\theta)_i + c_{13i}(\Delta\varepsilon_r)_i + c_{14i}(\Delta\varepsilon_\theta)_{i+1} + c_{15i}(\Delta\varepsilon_r)_{i+1} + c_{16i}(\Delta\varepsilon_z)] = \mu a^2\Delta p + \Delta f/\pi \tag{18}$$

where

$$c_{12i} = (r_{i+1}-r_i)r_i(d_{32})_i \quad , \quad c_{13i} = (r_{i+1}-r_i)r_i(d_{31})_i$$

$$c_{14i} = (r_{i+1}-r_i)r_{i+1}(d_{32})_{i+1} \quad , \quad c_{15i} = (r_{i+1}-r_i)r_{i+1}(d_{31})_{i+1}$$

$$c_{16i} = (r_{i+1}-r_i)[r_i(d_{33})_i + r_{i+1}(d_{33})_{i+1}] \tag{19}$$

Now we can form a system of 2n+3 equations for solving 2n+3 unknowns, $(\Delta\varepsilon_\theta)_i$, $(\Delta\varepsilon_r)_i$, at $i = 1,2,\ldots,n,n+1$ and $\Delta\varepsilon_z$. Equations (16), (17), and (18) are

taken as the first and last two equations, respectively, and the other 2n equations are set up at $i = 1,2,\ldots,n$ using Eqs. (10) and (13). The final system is an unsymmetric matrix of arrow type with the nonzero terms appearing in the last row and column and others clustering about the main diagonal, two below and two above. In the computer program which was developed, the dimensionless quantities r/a, $E\epsilon_r/\sigma_o$, $E\epsilon_\theta/\sigma_o$, $E\epsilon_z/\sigma_o$, σ_r/σ_o, σ_θ/σ_o, σ_z/σ_o, p/σ_o, q/σ_o, $f/(\pi a^2\sigma_o)$ were used in the formulation and the Gaussin elimination method was used to solve these equations. All calculations were carried out on IBM 4341 with double precision to reduce round-off errors.

OPTIMAL INCREMENTAL LOADING

Given any combination of incremental-loading (Δp, Δq, or Δf), we can now determine all incremental results (displacements, strains, and stresses) directly. No iteration is needed, while in Reference 4, many iterations in each step were required because a value for $\Delta\epsilon_z$ was assumed. The sizes of incremental-loading should be chosen properly in order to obtain accurate results at a reasonable cost. When the total applied pressure p is given, it is natural to divide the loading path in m equal fixed increments such as $\Delta p = (p-p^*)/m$. Larger values of m give more accurate results. A sequence of decreasing load-increments is a better choice than that of equal increments. In order to increase the efficiency without affecting the accuracy, an adaptive algorithm has been implemented on the basis of a scaled incremental-loading approach (ref 5).

In each step, a dummy load-increment such as Δp is applied and the incremental results $\Delta\sigma_i$ for $i = r,\theta,z$ at all grids are determined. For all

grid points at which $\sigma = ||\sigma_i|| < \sigma$, we compute the scaler α's by the formula

$$\alpha = \frac{1}{2} \{\Gamma + [\Gamma^2 + 4||\Delta\sigma_i||^2(\sigma_o{}^2 - ||\sigma_i||^2)]^{1/2}\}/||\Delta\sigma_i||^2 \tag{20}$$

where

$$\Gamma = ||\sigma_i||^2 + ||\Delta\sigma_i||^2 - ||\sigma_i+\Delta\sigma_i||^2 \tag{21}$$

and $||\sigma_i||$, $||\Delta\sigma_i||$, $||\sigma_i + \Delta\sigma_i||$ are computed by

$$||\sigma_i||^2 = \frac{1}{2} [(\sigma_r-\sigma_\theta)^2 + (\sigma_\theta-\sigma_z)^2 + (\sigma_z-\sigma_r)^2] \tag{22}$$

Let λ be the minimum of the α's. Then λ is the load-increment factor just sufficient to yield one additional point. A sequence of $\lambda(j)$ can be determined for all steps $j = 1,2,\ldots m$ and the updated results are

$$p^{(j)} = p^{(j-1)} + \lambda^{(j)}\Delta p^{(j)}$$

$$\sigma_i{}^{(j)} = \sigma_i{}^{(j-1)} + \lambda^{(j)}\Delta\sigma_i{}^{(j)} \quad , \quad \text{etc.} \tag{23}$$

This sequence of incremental loading is optimal for the present problem because all the coefficients c's in Eqs. (12), (14), and (19) are functions of the previous stresses and strains.

CONVERGENCE STUDY

In order to demonstrate the accuracy of the approach, four convergence studies are made. Consider a thick-walled tube of wall ratio $b/a = 2$ and subjected to internal pressure only. The cross-section of the tube is divided into n rings of equal thicknesses, i.e., $h = (b-a)/n$. The first problem is a closed-end tube loaded in the elastic range with $G = 10^5/3$ psi, $\nu = 0.3$, $p = 5$ psi. The numerical results with n = 10, 20, 50, 100 are shown in Table I together with the Lamé solution for the hoop stresses and strains at the boundaries a and b. The numerical results are correct up to four digits with n = 100.

TABLE I. ELASTIC SOLUTION FOR A CLOSED-END TUBE

($b/a = 2$, $G = 10^5/3$ psi, $\nu = 0.3$, $p = 5$ psi)

	n	$\sigma_\theta\|_a$	$\sigma_\theta\|_b$	$E\varepsilon_\theta\|_a$	$E\varepsilon_\theta\|_b$
Exact	-	8.3333	3.3333	9.3333	2.8333
FDM	10	8.3000	3.3000	9.3000	2.8000
	20	8.3256	3.3250	9.3250	2.8250
	50	8.3320	3.3320	9.3320	2.8320
	100	8.3330	3.3330	9.3330	2.8330

The second problem is the initial yielding solution for a plane-strain tube with $E/\sigma_o = 200$, $\nu = 0.3$, $\varepsilon_z = 0$. The numerical results with n = 10, 20, 50, 100, 200, are shown in Table II together with the exact solution for the dimensionless $\bar{p} = p/\sigma_o$, $\bar{\sigma}_\theta = \sigma_\theta/\sigma_o$, $\bar{\varepsilon}_\theta = (E/\sigma_o)(\varepsilon_\theta)$ at $r = a$ and b.

TABLE II. INITIAL YIELDING SOLUTION FOR A PLANE-STRAIN TUBE

($b/a = 2$, $E/\sigma_o = 200$, $\nu = 0.3$)

	n	$\bar{p}$	$\bar{\sigma}_\theta\|_a$	$\bar{\sigma}_\theta\|_b$	$\bar{\varepsilon}_\theta\|_a$	$\bar{\varepsilon}_\theta\|_b$
Exact	-	.43229	.72049	.28820	.82424	.26226
FDM	10	.00110	-.00107	-.00216	-.00054	-.00065
	20	.00028	-.00027	-.00054	-.00014	-.00016
	50	.00005	-.00004	-.00009	-.00002	-.00003
	100	.00001	-.00001	-.00003	-.00001	-.00001
	200	.00001	.00000	-.00001	.00000	.00000

The third problem is the elastic-perfectly plastic solution for a plane-strain tube with $b/a = 2$, $E/\sigma_0 = 200$, $\nu = 0.3$, $\omega = 0$, $\varepsilon_z = 0$. The numerical results with n = 10, 20, 50, 100, 200 are shown in Table III for the pressure and displacement at the bore corresponding to 50 percent and 100 percent overstrain, i.e., $\rho/a = 1.5$ and 2.0.

TABLE III. ELASTIC-PERFECTLY PLASTIC SOLUTION FOR A PLANE-STRAIN TUBE

($b/a = 2$, $E/\sigma_0 = 200$, $\nu = 0.3$)

	50% Overstrain		100% Overstrain	
n	$\bar{p}$	$\bar{\varepsilon}_\theta\vert_a$	$\bar{p}$	$\bar{\varepsilon}_\theta\vert_a$
10	.71863	1.97725	.79849	3.68937
20	.71825	1.96969	.79790	3.66831
50	.71808	1.96780	.79760	3.66280
100	.71803	1.96759	.79751	3.66211
200	.71801	1.96757	.79747	3.66199

The numerical results converge and are accurate up to four digits with n = 100. There is no closed-form solution available for comparison. The famous paper by Hodge and White (ref 2) has been used quite often as a basis for assessing the accuracy of other approximate methods. However, the numerical integration is cumbersome. The present formulation is much simpler and seems more accurate. In order to further demonstrate the accuracy of the present approach, a convergence study for an incompressible, ideally-plastic thick tube in plane-strain condition has been made and compared with exact solution (ref 2). The numerical results for a nearly incompressible material (ν =

0.49999) are shown in Table IV together with the exact solution ($\nu = 1/2$) for the internal pressure and the displacement at the bore corresponding to 50 percent and 100 percent overstrain.

TABLE IV. INCOMPRESSIBLE, IDEALLY-PLASTIC SOLUTION FOR A PLANE-STRAIN TUBE ($b/a = 2$, $E/\sigma_\theta = 200$, $\nu = 0.49999$)

	50% Overstrain		100% Overstrain	
n	$\bar{p}$	$\bar{\epsilon}_\theta\|_a$	$\bar{p}$	$\bar{\epsilon}_\theta\|_a$
10	.72069	1.95714	.80015	3.48806
20	.72077	1.95072	.80034	3.47012
50	.72078	1.94891	.80038	3.46509
100	.72078	1.94865	.80038	3.46437
Exact	.72078	1.94856	.80038	3.46410

We may thus conclude that exact solutions can be obtained by this numerical approach.

ADDITIONAL RESULTS

After establishing the convergence and accuracy of this new approach, the numerical results for more general problems have been obtained. Some of the additional results are documented here for future comparison by others. All numerical results presented here are for a thick-walled tube with wall ratio $b/a = 2$, $E/\sigma_o = 200$, $\nu = 0.3$, $n = 100$, $\omega = E_t/E = 0.1$. The numerical results are accurate up to four or five digits. Table V shows the results of the dimensionless quantities p/σ_o, σ_θ/σ_o, σ_z/σ_o, $(E/\sigma_o)\epsilon_r$, $(E/\sigma_o)\epsilon_\theta$, $(E/\sigma_o)\epsilon_z$ at

the inside or elastic-plastic boundary ρ for $\rho/a = 1.0, 1.1, 1.2, \ldots, 2.0$ in a plane-strain tube with strain-hardening parameter $\omega = 0.1$. Tables VI and VII show the similar results of the dimensionless stresses, strains, displacement at the bore or elastic-plastic boundary for various stages of elastic-plastic loadings in an open-end or closed-end tube, respectively.

TABLE V. ELASTIC-PLASTIC SOLUTION FOR A PLANE-STRAIN TUBE

($b/a = 2$, $E/\sigma_o = 200$, $\nu = 0.3$, $E_t/E = 0.1$, $n = 100$, $\varepsilon_z = 0$)

ρ/a	p/σ_o	Inside σ_θ/σ_o	$\sigma_\theta/\sigma_o\|_\rho$	Inside σ_z/σ_o	$\sigma_z/\sigma_o\|_\rho$	$\frac{E}{\sigma_o}\varepsilon_r\|_a$	$\frac{E}{\sigma_o}\frac{U_a}{a}$
1.0	.43229	.72049	.72049	.08646	.08646	−.67438	.82424
1.1	.51296	.66972	.75016	.05586	.10453	−.91636	1.00141
1.2	.58362	.63049	.78249	.02537	.12427	−1.17381	1.20371
1.3	.64556	.60225	.81739	−.00409	.14566	−1.44917	1.43022
1.4	.69982	.58427	.85479	−.03148	.16866	−1.73882	1.68000
1.5	.74725	.57572	.89457	−.05589	.19323	−2.04304	1.95207
1.6	.78851	.57579	.93667	−.07658	.21932	−2.36107	2.24535
1.7	.82416	.58376	.98092	−.09302	.24687	−2.69205	2.55869
1.8	.85466	.59898	1.02718	−.10490	.27581	−3.03504	2.89081
1.9	.88041	.62086	1.07530	−.11205	.30606	−3.38898	3.24034
2.0	.90177	.64885	1.12509	−.11447	.33753	−3.75274	3.60578

TABLE VI. ELASTIC-PLASTIC SOLUTION FOR AN OPEN-END TUBE

($b/a = 2$, $E/\sigma_o = 200$, $\nu = 0.3$, $E_t/E = 0.1$, $n = 100$)

ρ/a	p/σ_o	$\frac{\sigma_\theta}{\sigma_o}\vert\rho$	$\frac{\sigma_\theta}{\sigma_o}\vert\rho$	$\frac{\sigma_z}{\sigma_o}\vert a$	$\frac{\sigma_z}{\sigma_o}\vert\rho$	$\frac{E}{\sigma_o}\,\varepsilon_r\vert a$	$\frac{E}{\sigma_o}\,\frac{U_a}{a}$	$\frac{E}{\sigma_o}\,\varepsilon_z$
1.0	.42857	.71429	.71429	0.0	0.0	-.64286	.84286	-.08571
1.1	.50697	.66772	.74089	-.02818	.00090	-.86384	1.01920	-.10234
1.2	.57515	.63143	.76928	-.05498	.00334	-1.09744	1.21679	-.11883
1.3	.63434	.60489	.79916	-.08033	.00689	-1.34207	1.43368	-.13552
1.4	.68546	.58744	.83021	-.10392	.01106	-1.59572	1.66769	-.15275
1.5	.72929	.57829	.86195	-.12542	.01529	-1.85584	1.91616	-.17089
1.6	.76641	.57659	.89385	-.14463	.01895	-2.11934	2.17590	-.19034
1.7	.79730	.58141	.92520	-.16147	.02135	-2.38245	2.44299	-.21149
1.8	.82236	.59175	.95517	-.17604	.02181	-2.64070	2.71271	-.23466
1.9	.84191	.60652	.98280	-.18862	.01965	-2.88902	2.97950	-.26008
2.0	.85630	.62457	1.00708	-.19963	.01432	-3.12192	3.23718	-.28781

TABLE VII. ELASTIC-PLASTIC SOLUTION FOR A CLOSED-END TUBE

($b/a = 2$, $E/\sigma_o = 200$, $\nu = 0.3$, $E_t/E = 0.1$, $n = 100$)

ρ/a	p/σ_o	$\frac{\sigma_\theta}{\sigma_o}\|\rho$	$\frac{\sigma_\theta}{\sigma_o}\|\rho$	$\frac{\sigma_z}{\sigma_o}\|a$	$\frac{\sigma_z}{\sigma_o}\|\rho$	$\frac{E}{\sigma_o}\epsilon_r\|a$	$\frac{E}{\sigma_o}\frac{U_a}{a}$	$\frac{E}{\sigma_o}\epsilon_z$
1.0	.43301	.72169	.72169	.14434	.14434	-.69282	.80829	.05774
1.1	.51408	.66902	.75199	.11139	.17341	-.94569	.98359	.06863
1.2	.58514	.62897	.78518	.07731	.20331	-1.21580	1.18564	.07861
1.3	.64758	.60074	.82124	.04394	.23449	-1.50300	1.41370	.08814
1.4	.70247	.58332	.86014	.01276	.26717	-1807142	1.66712	.09746
1.5	.75069	.57573	.90186	-.01509	.30145	-2.12799	1.14532	.10665
1.6	.79290	.57710	.94636	-.03887	.33724	-2.46525	2.24773	.11566
1.7	.82963	.58672	.99359	-.05811	.37437	-2.81846	2.57373	.12432
1.8	.86323	.60610	1.04858	-.07381	.41641	-3.22466	2.95874	.13314
1.9	.88833	.62834	1.09583	-.08243	.45136	-3.56994	3.29352	.13946
2.0	.91091	.65936	1.15052	-.08775	.49023	-3.96597	3.68518	.14507

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